Inflation Risk and Asset Market Disturbances: The Mean-Variance Model Revisited

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This paper empirically re-investigates the international CAPM model. In contrast to previous studies by Frankel (1982) and Frankel and Engel (1984), the estimation model does not require that there be no asset market disturbances. Also, rather than requiring purchasing power parity or that the only source of inflation uncertainty arise from a weighted average of exchange rate changes as in the previous studies, inflation risk is simply measured as the uncertain component in the consumer price index of each country. To allow for these additional sources of uncertainty, an alternative estimation method is derived to identify the covariance constraints.

The poor forecasting ability of the forward exchange rate as a predictor of the future spot rate has emerged as an empirical regularity of the floating rate period. A risk premium in the forward exchange rate would explain this behavior.[1] Mean-variance models have been estimated as a way to empirically investigate the risk premium. Frankel (1982) and Frankel and Engel (1984), for example, impose restrictions from a mean-variance portfolio model to identify the parameter of relative risk aversion, finding that they cannot reject levels of the parameter of relative risk aversion over a relatively large range that includes zero. To identify the model, they must assume that there are no asset market disturbances and they require specific behavior from price levels. That is, either purchasing power parity must hold, or else the random component of prices must arise from a weighted average of nominal exhange rates.

This paper investigates the presence of a risk premium using a mean-variance optimizing model, allowing for disturbances to the asset market model and explicit deviations from purchasing power parity. In contrast to previous studies, the estimation methodology does not place restrictions upon the form of price

inflation. Instead, inflation is simply measured from consumer price indices by country. The estimation method accomplishes these generalizations by introducing an alternative way to identify the covariance constraints. The alternative method requires specifying an information set that the market uses to form forecasts. The results in the paper are compared across different information sets.

The plan of the paper is the following. Section I describes the estimation method and how it differs from previous studies. Section II reports the empirical results. Concluding remarks follow.

I. Estimation Model and Method

As a potential explanation of a risk premium in the foreign exchange market, the international mean-variance model has been used to investigate the coefficient of relative risk aversion. Theoretical studies of the international portfolio models include Kouri (1977), Hodrick (1981), Stulz (1981), and Adler and Dumas (1983). Empirically, Frankel (1982) and Frankel and Engel (1984), among others, have studied the international asset market using the mean-variance approach. Since these studies have found a large range of possible values for the parameter of risk aversion, both risk neutrality and levels implying relatively strong risk aversion cannot be rejected.

As will be described below, the estimation method employed in these empirical studies constrains the number of covariance restrictions according to the number of asset demand equations. Due to this constraint, the estimation method has required that additional restrictions be placed upon the source of inflation risk. The studies have either required purchasing power parity (*e.g.*, Frankel and Engel, 1984) or else that inflation is not stochastic except for exchange rate uncertainty (*e.g.*, Frankel, 1982). Additionally, in order to identify the covariance restrictions, these studies use an estimation method that requires the asset market demand equations to hold exactly and are not subject to disturbances of their own. As the asset market under study is the market for total world outside assets, this assumption may be strong.

By contrast, the empirical analysis below relaxes these assumptions by investigating an international mean-variance asset market model that both allows the demand equations to contain disturbances and does not restrict the form of country-specific inflation risk. Allowing for these disturbances requires a different method of identifying the covariance restrictions. Section I.A describes the estimation model and compares it with Frankel (1982) and Frankel and Engel (1984). Section I.B discusses the method of estimation.

I.A. Inflation Risk and the Estimation Model

The international mean-variance model posits that an investor will diversify the currency denomination of his wealth according to the covariance of the asset returns with the sources of risk in the portfolio. He allocates the currency denomination of his risk in this way in order to trade off the variance and the expected return of his portfolio according to his degree of risk tolerance. Since the mean-variance model has been analyzed theoretically elsewhere, only the basic framework of the model and its results will be treated below, leaving the derivation of the present model to the appendix.

Assume that there are k different countries in the world, each country having a distinct price index that measures the cost of the consumption bundle of its residents. As the inflation rates differ across countries, the real rate of return relevant to each investor depends upon his country of residence. Residents hold interest-bearing assets denominated in foreign currency as well as their own. Since there are k currencies, there are $k-1$ independent types of these assets differentiated by currency. From the point of view of an investor living in country h, the real rates of return on assets denominated in currency i can be written in the following form (using the approximation in Frankel, 1982):[2]

$$\langle 1 \rangle \qquad 1 + \mu_{t+1}^{i,h} = \frac{(1+r_t^i)}{(1+\pi_t^h)(1+D(s_t^i))} \approx 1 + r_t^i - D(s_t^i) - \pi_t^h$$

$$\text{for } i = 1, \ldots, k-1;$$

$$1 + \mu_{t+1}^{i,h} = \frac{(1+r_t^i)}{(1+\pi_t^h)} \approx 1 + r_t^* - \pi_t^h, \qquad \text{for } i = k;$$

where $\mu^{i,h}$ is the real rate of return of an asset denominated in currency i for an investor in country h; r^i is the nominal rate on the ith currency asset, r^* is the nominal rate on the numeraire currency bond, π^h is the inflation rate of the hth country measured in terms of the numeraire currency, s^i is the logarithm of the ith currency price of a numeraire currency unit, and $D(q_t) \equiv q_{t+1} - q_t$ (for any variable q) is the forward difference operator.

Define μ_t^h and x_t^h as, respectively, the $k-1$ dimensional vectors of the real rates and the vector of asset shares by currency denomination held by investors in country h. Then the next period real wealth to h country residents can be written,

$$\langle 2 \rangle \qquad W_{t+1}^h = W_t^h + W_t^h[x_t^{h\prime}\mu_{t+1}^h + (1 - x_t^{h\prime}\iota)\mu_{t+1}^{*,h}]$$

where $\mu_t^{*,h}$ is the scalar real return on the numeraire currency asset and ι is a $k-1$ component of ones.

Assuming that inflation and exchange rates are uncertain, and that current nominal interest rates are known, the expected value and variance of next period's wealth conditional on current information can be written,

$$\langle 3 \rangle \qquad E_t W_{t+1}^h = W_t^h(1 + r_t^* + x_t^{h\prime}(r_t - \iota r_t^*)) - W_t^h(x_t^{h\prime}E_t D(s_t) + E_t\pi_{t+1}^h),$$

$$\langle 4 \rangle \qquad E_t(W_{t+1}^h - EW_{t+1}^h)^2 = (W_t^h)^2 E_t[x_t^{h\prime}D^*(s_t) + (\pi_{t+1}^h - E_t\pi_{t+1}^h)]^2,$$

where E_t is the expectation operator conditional on time t information, s_t is the vector of the logarithm of exchange rates, r_t is the vector of nominal interest rates, and $D^*(q_t) \equiv D(q_t) - E_t D(q_t)$, the forecast error (for arbitrary random variable, q_t). For future reference, it will prove convenient to define: $E_t q_{t+1} - q_{t+1} \equiv \varepsilon_{t+1}^q$, so that $D^*(q_t) = \varepsilon_{t+1}^q$.

Investors in each of the k countries are assumed to maximize an identical utility function with constant relative risk aversion. Aggregating the asset demand equations of the individual countries gives the total world demand for assets denominated in different currencies. To investigate the risk premium, the aggregated asset demand equations are inverted so that the asset supplies determine the relative rates of return across countries. As in Frankel (1982) and Frankel and Engel (1984), the variances of innovations are assumed to be constant.[3] Defining w_t as the vector of country shares in world wealth, these rates of return can be written

as a function of the asset supplies and the distribution of world wealth across countries:

$$\langle 5 \rangle \qquad\qquad r_t - tr_t^* - E_t D(s_t) = \rho(\Omega x_t + \Gamma w_t),$$

where

$$\Omega = E[\varepsilon_{t+1}^i \varepsilon_{t+1}^i{}'], \qquad (k-1) \times (k-1),$$

$$\Gamma = E[\varepsilon_{t+1}^i \varepsilon_{t+1}^\pi{}'], \qquad (k-1) \times k,$$

ρ is the parameter of relative risk aversion, Ω is the variance–covariance matrix of exchange rate forecast errors, and Γ is the matrix of covariances between exchange rate and inflation rate forecast errors, measured in terms of the numeraire currency.

The relative rates of return in equation $\langle 5 \rangle$ depend upon two components. The first component in equation $\langle 5 \rangle$ arises from the nominal risk associated with holding assets denominated in different currencies. It, therefore, depends upon the covariance of nominal exchange rate forecast errors, Ω, and the supply of outside assets, x_t. Since the nominal exchange rate risk incurred from asset holdings is independent of country residence, the nominal covariance relationship is common to all of the country-specific demand equations.

On the other hand, the covariance relationship in the second component of equation $\langle 5 \rangle$ arises from hedging the risk associated with the uncertain inflation rate in each country. The covariances between inflation and exchange rates reflect the effects of the country-specific inflation uncertainty upon the equilibrium relative rates of return. In this inverted asset demand system, the covariances between the inflation rates of each country h and the exchange rates are given by the hth column of Γ. The components of this vector, Γ^h, are weighted by each country's share in world wealth, w_t^h.

Purchasing power parity presents an interesting special case. Under PPP, the *inflation rate of each country measured in terms of the numeraire currency is simply the numeraire inflation rate:* $\pi_t^* = \pi_t^h$ for all $h = 1, \ldots, k-1$. The columns of Γ are identical in this case and equal to $E[\varepsilon_{t+1}^i \varepsilon_{t+1}^{\pi*}]$. By substituting this restriction into equation $\langle 5 \rangle$, the inverted portfolio reduces to:

$$\langle 6 \rangle \qquad\qquad r_t - tr_t^* - E_t D(s_t) = \rho(\Omega x_t + E[\varepsilon_{t+1}^i \varepsilon_{t+1}^{\pi*}]),$$

where $\varepsilon_{t+1}^{\pi*} \equiv \pi_{t+1}^* - E_t \pi_{t+1}^*$.

Since this model resembles those estimated by Frankel (1982) and Frankel and Engel (1984), the models in each of these papers provide an interesting comparison. First, Frankel (1982) imposes the assumption that the price level can be measured as the consumption-weighted average of the prices of individual goods. Each good is assumed to be produced in a particular country where the price of the good is not stochastic. Under this assumption, the only uncertain components of the price level arise from changes in the exchange rates that, in turn, alter the cost of purchasing foreign goods in terms of the country's own currency.

Specifically, Frankel (1982) assumes that the inflation rate (measured in terms of the numeraire) is given by:

$$\langle 7 \rangle \qquad\qquad \pi_t^h = \alpha^h{}'(\pi^c - D(s_t)) + (1 - \alpha^h{}' \iota)\pi^{c*},$$

where π^c is the vector with typical component equal to the rate of price changes in the commodity of country i measured in terms of the ith currency, and π^{c*} is the

(scalar) price change of the commodity produced by the numeraire country. Also, α^b is a $(k-1)$ dimensional vector of consumption shares of commodities consumed by country b residents, and $(1 - \alpha^{b\prime}\iota)$ is the share of the numeraire country's good in consumption. Furthermore, since Frankel (1982) assumes the commodity prices are known, the forecast error in inflation reduces to:

$$\langle 8 \rangle \qquad \qquad \pi^b_{t+1} - E_t \pi^b_{t+1} = -\alpha^{b\prime} \varepsilon^s_{t+1}.$$

Substituting $\langle 8 \rangle$ into the definition of Γ following equation $\langle 5 \rangle$, implies that the covariance between exchange rate and inflation rate errors can be written solely in terms of the exchange rate covariance matrix and the consumption shares, i.e., $\Gamma = -\Omega\alpha$. In this special case, the system of inverted asset demand equations in equation $\langle 5 \rangle$ above have the following form,

$$\langle 9 \rangle \qquad \qquad r_t - \iota r^*_t - E_t D(s_t) = \rho \Omega(x_t - \alpha w_t),$$

where α is the $(k-1) \times k$ matrix with typical row equal to the consumption shares across countries for a particular country.[4]

The reason for the particular assumption about the structure of the inflation rates arises from the unobserved component of the left-hand-side variable. In equation $\langle 9 \rangle$, the expected relative rates of return depend upon the expected change in the exchange rate, an unknown variable. But under the assumption of rational expectations, the conditional expectation of the exchange rate can be replaced by the actual *ex post* exchange rate plus a white noise error term: $E_t D(s_t) = D(s_t) + \varepsilon^s_{t+1}$. Making this substitution in equation $\langle 9 \rangle$ gives Frankel's estimating equation (equation $\langle 11 \rangle$ in his paper):

$$\langle 10 \rangle \qquad \qquad r_t - \iota r^*_t - D(s_t) = \rho \Omega(x_t - \alpha w_t) + \varepsilon^s_{t+1}.$$

Assuming that the asset demand equations hold exactly implies that the cross-equation covariances of the residuals, ε_t, are the same as the covariance parameters that affect asset demand, Ω. Also, specifying that the random component of price levels occurs only from exchange rate innovations requires identifying the covariances of exchange rate changes only. He identifies these parameters by constraining the covariance of the residuals of equation $\langle 9 \rangle$ to be the parameters on the asset supplies.

As a second comparison, Frankel and Engel (1984) use this empirical technique to estimate a different form of the international asset market model. Instead of using the approximation of real rates of return given in $\langle 1 \rangle$ above, they measure the real rates for investors of all countries as:

$$\langle 11 \rangle \qquad \qquad (1 + \mu^i_{t+1}) \equiv \frac{(1 + r^i_t)}{(P^*_{t+1}/P^*_t)(S^i_{t+1}/S^i_t)},$$

where $P^* \equiv \prod_{b=1}^{k} (P^b S^b)^{\alpha b}$ for P^b the (CPI) price level of country b, and S^b, the level of the nominal exchange rate level of country b (equal to 1 for $b=k$). Thus, the prices are again assumed to be a consumption-share weighted average of the 'goods prices' from each country, where these 'goods prices' are measured by the consumer price indices of each country. To treat these real rates of return as the left-hand-side variables, Frankel and Engel (1984) must assume purchasing power parity —that the consumption shares are the same across countries —so that the real rates of return are the same across countries except for exchange rate changes.[5]

Using the same steps as in the Frankel model, this version of the model can be written in terms of real interest differentials employing the rational expectations assumption (given as equation $\langle 6 \rangle$ in Frankel and Engel, 1984):

$$\langle 12 \rangle \qquad E_t(\mu_{t+1} - \iota\mu_{t+1}^*) = \rho(E[\varepsilon_{t+1}^{\prime\prime}\varepsilon_{t+1}^{\prime\prime}{}']x_t + E[\varepsilon_{t+1}^{\prime\prime}\mu_{t+1}^*]),$$

where $E_t(\mu_{t+1} - \iota\mu_{t+1}^*) = (\mu_{t+1} - \iota\mu_{t+1}^*) + \varepsilon_{t+1}^{\prime\prime}$. Due to the differences in approximating real rates of return, the basic model derived above cannot be directly reduced to the equations to their model. But clearly, the Frankel–Engel model in equations $\langle 12 \rangle$ and the form of the model under purchasing power parity given in equations $\langle 6 \rangle$ share a strong resemblance. In both cases, the asset supplies affect the relative rates of return according to the covariances of the surprises in real interest rates. And both equations contain a constant that is the covariance between these surprises and the innovations in the numeraire asset.

Using the assumption of rational expectations, Frankel and Engel substitute the actual computed relative rates of return for the actual rates on the left-hand side. Imposing the condition that the asset demand equations hold exactly, the residual in the equation becomes $\varepsilon_{t+1}^{\prime\prime}$ so that, as before, the coefficients on the asset supplies may be constrained using the covariance matrix of the residuals. Because Frankel and Engel (1984) only have estimates of the covariance parameters to asset demand that can be identified from the covariance matrix of residuals, they cannot identify the covariances between the relative real rates of return and the real rate of return on the numeraire asset. As a result, they cannot impose the covariance constraints upon the constant coefficients, as implied by their model given in equation $\langle 12 \rangle$.

Both of these examples serve to illustrate that using this estimation method to identify the covariance restrictions from mean-variance optimization restricts the source of risk that may be investigated. Since the covariance matrix of residuals identifies the covariance matrix of random components that affects asset demand, this method requires forms of the model in which the forecast errors on the left-hand side are the only sources of uncertainty in the portfolio. Therefore, identifying the covariance restrictions for more general forms of the model as in equation $\langle 5 \rangle$ requires a method that will allow for country-specific inflation uncertainty as well as nominal exchange rate uncertainty.

I.B. *The Estimation Methodology*

The following method allows for both general price uncertainty measured from consumer price levels and disturbances to asset demand. Under rational expectations, the projection of the changes in exchange rates and prices on the market's information set provides one-step ahead forecast errors that are orthogonal to that information set. Thus, if z_t is the vector of components of the information set added at time t, this projection can be written as:

$$\langle 13 \rangle \qquad \begin{vmatrix} D(s_t) \\ \pi_{t+1} \end{vmatrix} = \theta(L)z_t + \varepsilon_{t+1},$$

where $\theta(L)$ is a matrix of polynomials in the lag operator and ε_t is a vector where the first $(k-1)$ components are exchange rate forecast errors, ε', and the next k components are inflation forecast errors, ε^π. Partitioning the contemporaneous

covariance matrix of ε identifies the unobserved parameters, Ω and Γ.

$$\langle 14 \rangle \qquad E(\varepsilon_{t+1}\varepsilon_{t+1}') = \begin{bmatrix} \Omega & \Gamma \\ \Gamma' & Q_{33} \end{bmatrix}.$$

By using these equations to identify the innovations to the forecasts of the exchange rate and inflation rate processes, the covariance parameters of each underlying source of uncertainty may be considered separately.

In addition, the asset market equations in $\langle 5 \rangle$ are allowed to contain disturbances. The errors to the asset market equations may embody structural shocks from the country-specific asset demand functions. These structural shocks may arise, for example, from disturbances from other markets not captured in the asset market model. Defining the $k-1$ dimensional vector of these disturbances as ψ_t, the asset demand equations can be rewritten,

$$\langle 15 \rangle \qquad r_t - r_t^* \imath - E_t D(s_t) = \rho \Omega x_t + \rho \Gamma w_t + \psi_t,$$

where the asset market disturbances are assumed independent and identically distributed with covariance matrix Q_{11}; i.e., $\psi_t \sim N(0, Q_{11})$ for all t.[6]

Substituting the rational expectations forecasts for the unobserved terms into the portfolio equations identifies the parameter of risk aversion, ρ. Combining the asset market equations in $\langle 15 \rangle$ with the assumption of rational expectations in $\langle 13 \rangle$ and imposing the covariance constraint $\langle 14 \rangle$ gives the following system of equations written in block form.

$$\langle 16 \rangle \qquad \begin{bmatrix} r_t - \imath r_t^* \\ D(s_t) \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \theta_1(L) \\ \theta(L) \end{bmatrix} z_t + \rho \begin{bmatrix} \Omega & \Gamma \\ & 0 \end{bmatrix} \begin{bmatrix} x_t \\ w_t \end{bmatrix} + \begin{bmatrix} \psi_t \\ \varepsilon_{t+1}^s \\ \varepsilon_{t+1}^\pi \end{bmatrix}$$

$$\Omega = E(\varepsilon_{t+1}^s \varepsilon_{t+1}^s{}')$$

such that:

$$\Gamma = E(\varepsilon_{t+1}^s \varepsilon_{t+1}^\pi{}'),$$

where $\theta_1(L)$ is the matrix comprised of the top $(k-1)$ rows of $\theta(L)$ corresponding to the exchange rate forecasts. 0 is a conformable matrix with all components equal to zero.

As shown in equations $\langle 16 \rangle$, the structure of expectations leads to a stacked system of equations. The first block is the $(k-1)$ dimensional system of asset market equations. In the second and third blocks are the $(k-1)$ exchange rate forecasting equations and the k inflation forecasting equations, respectively. The system of equations identifies the parameter of interest, ρ, by imposing the constraint that the relevant block of the covariance matrix of residuals are the parameters, Ω and Γ, in the asset market equations. As shown in $\langle 14 \rangle$, the covariance matrix of innovations to the exchange rate forecasting equations is Ω, while the matrix of covariances of these errors with the inflation forecast errors define Γ.

Therefore, this method both explicitly allows for deviations from PPP and does not require estimated asset market equations to hold exactly. On the other hand, it requires taking a stand on the information set that agents use to form their inflation and exchange rate forecasts. For this reason, different information sets are used in

the following section to check the sensitivity of the results to the particular information set.

II. Estimation and Results

The estimation used asset market data for six different countries —Canada, France, Germany, Japan, the United Kingdom, and the United States —so that $k=6$. The monthly data cover the period from January 1975 through December 1981. Prior to this period, many of the series required for construction of the bond supplies are not available. The asset series measure the sum of money and the stock of official debt from all governments that are denominated in a particular currency.[7] With the exception of France, the data series for each country are described in Lewis (1988). The data series for France are modified from Frankel (1982). A detailed data appendix is available from the author upon request.

Another important consideration in estimating the model is the choice of interest rate series. Ideally, one would want to have an interest rate which accurately reflected the rate of return on the sum total of official debt of all governments that is denominated in a particular currency. The differentials in the rates of return among assets should measure differences that arise due to currency preference alone and not to other factors such as maturity, capital controls, or default risk. For this reason, interest rates on one-month Eurocurrency deposits were chosen as measures of the rates of return on the assets.

Since the results are potentially sensitive to the information set, the model was estimated using two different information sets. The first contained lagged values of the first differences of exchange rates and consumer price indices. The second information set included money and income as well (where money supply was measured as M1 or the foreign counterpart, and income was proxied by industrial production). The forecasting autoregressions, $\theta(L)$, were truncated using likelihood ratio lag length tests.[8] For both information sets, the length was truncated at lag two.

When the number of independent currencies is small, the system of equations ⟨16⟩ can be estimated jointly with maximum likelihood.[9] But since the number of equations is $3k-2$ ($k-1$ asset equations, $k-1$ exchange rate equations and k price equations), the size of the system increases very rapidly with the number of countries. The unusual covariance constraints further compound the computational burden since the number of parameters in Ω and Γ rise with k. Therefore, in order to allow for a reasonable number of currencies without jeopardizing computational tractability, the 'one-step' maximum likelihood method was used.[10] In addition, some simplifying assumptions were made to shrink the size of the information matrix. In particular, it was assumed that the forecast errors and asset market disturbances are orthogonal, $E(\varepsilon_{t+1}\psi_t)=0$, a condition that holds under rational expectations if the asset supplies are components in the information set of agents at time t. The details of the one-step method applied to this problem are given in the statistical appendix.

The one-step maximum likelihood method requires an initial consistent estimate of the parameters. In the present case, these initial estimates were formed using the following two-stage procedure. In the first stage, the vector of exchange rate changes and inflation were regressed on the information sets as in equation ⟨13⟩. The covariance matrix of residuals to these regressions provided estimates of the

TABLE 1. Initial parameter estimates based upon information set of lagged exchange rates and inflation rates.

| Exchange rate | 2Ω: Covariance matrix of exchange rates (×2) | | | | |
	Canadian dollar	British pound	French franc	German DM	Japanese yen
Canadian dollar	3.8	1.0	1.0	1.2	0.5
British pound	1.0	17.0	8.1	8.2	8.2
French franc	1.0	8.1	15.1	14.3	7.9
German DM	1.2	8.2	14.3	18.4	7.4
Japanese yen	0.5	8.2	7.9	7.4	18.7

| Exchange rate | 2Γ: Covariance matrix of exchange rates and inflation (×2) | | | | | |
	Canadian inflation	British inflation	French inflation	German inflation	Japanese inflation	US inflation
Canadian dollar	−3.8	−0.9	−0.8	−1.3	−0.5	0.1
British pound	−0.8	−17.0	−7.8	−8.1	−7.6	0.1
French franc	−0.9	−8.7	−15.0	−14.3	−8.3	0.0
German DM	−1.1	−8.6	−14.4	−18.3	−7.8	0.1
Japanese yen	−0.7	−8.0	−7.9	−7.1	−18.9	−0.0

Estimated parameter of relative risk aversion $\rho = -2.9$.

covariance matrices as in equations ⟨14⟩. In the second stage, these estimates of $\hat{\Omega}$ and $\hat{\Gamma}$ were imposed upon the top block of the system to obtain a consistent estimate of $\hat{\rho}$.

Tables 1 and 2 present the initial consistent estimates for the covariance and risk aversion parameters using each information set. Since Frankel and Engel (1984) (as well as Frankel, 1982) report estimates of $\rho\Omega$ with ρ constrained to be 2, the covariance parameters were multiplied by 2 to allow for comparison.[11] Although this method of identifying the covariances is different than the Frankel and Engel method, the estimates of the Ω matrix are quite similar. For these exchange rate covariance parameters in both of the information sets in Tables 1 and 2, the estimates are roughly the same as those reported in Table 2 of Frankel and Engel. Generally, the diagonal estimates here are about the same as the Frankel–Engel estimates although slightly smaller in scale.

However, since Frankel and Engel (1984) do not allow for deviations from PPP, their model does not allow assets to hedge country-specific inflation risk through the covariances between exchange rate and inflation rate errors given in Γ. As the tables indicate, these estimates are primarily negative, arising largely from changes in the exchange rate used to convert domestic prices into the numeraire currency. Also, the initial estimates of the parameters of risk aversion differ somewhat more across information sets; for the lagged exchange rate and inflation rate information set, the estimate is negative.

To find the efficient estimates, these initial parameter estimates were adjusted by the asymptotic information matrix formed by computing the hessian of the likelihood function. The covariance parameters were essentially unchanged from Tables 1 and 2. The effects of the 'step' upon the estimate of ρ depended upon the

TABLE 2. Initial parameter estimates based upon information set of lagged exchange rates, inflation rates, money supplies and income.

| Exchange rate | 2Ω: Covariance matrix of exchange rates ($\times 2$) | | | | |
	Canadian dollar	British pound	French franc	German DM	Japanese yen
Canadian dollar	3.0	1.0	1.0	1.1	0.9
British pound	1.0	14.3	6.2	6.7	7.1
French franc	1.0	6.2	12.1	11.7	6.2
German DM	1.1	6.7	11.7	15.5	6.5
Japanese yen	0.9	7.1	6.2	6.5	15.6

| Exchange rate | 2Γ: Covariance matrix of exchange rates and inflation ($\times 2$) | | | | | |
	Canadian inflation	British inflation	French inflation	German inflation	Japanese inflation	US inflation
Canadian dollar	-3.1	-1.2	-0.9	-1.2	-1.1	0.0
British pound	-0.9	-14.2	-5.9	-6.6	-6.7	0.1
French franc	-1.0	-7.0	-12.1	-11.8	-6.8	-0.0
German DM	-1.0	-7.5	-11.9	-15.5	-7.0	0.1
Japanese yen	-1.3	-6.5	-6.4	-6.2	-15.9	-0.1

Estimated parameter of relative risk aversion $\rho = 6.1$.

information set. For the first information set of lagged inflation and exchange rates, the 'step' moved the parameter estimate very slightly in the positive direction, a change imperceptible at the order of rounding. The 'step' from the expanded information set including money and income moved the estimate of ρ down to a value of 5.3. But parameter estimates of ρ over a very large range could also not be ruled out. For both information sets, a value of ρ equal to zero could not be rejected. This result concurs with the findings of the previous studies. Frankel and Engel report their maximum likelihood estimate of ρ at -67, but cannot reject a value equal to zero. Also, Frankel (1982) reports that his estimated model cannot reject values of ρ that include zero as well as near 30. Although the point estimates here are closer to zero than those reported in the other two studies, the estimated model cannot reject values of ρ over a range similar to the other studies.

To investigate the restrictions of the model, the two-step estimation constraints were tested using a log likelihood ratio test.[12] Similar to the results found by the other studies, the covariance constraints were rejected at the 99 per cent confidence level. While the critical value for the chi-squared statistic with 54 degrees of freedom is about 82, the statistic obtained from the model using the information set of lagged prices and exchange rates was 205; for the expanded information set, it was 142. These rejections accord with the results of both Frankel (1982) and Frankel and Engel (1984), where the over-identifying restrictions were rejected at high confidence levels.

III. Concluding Remarks

In this paper, the presence of a foreign exchange risk premium was investigated

using a mean-variance optimizing model in which inflation risk was explicitly introduced without requiring purchasing power parity. Also, the empirical method developed in this paper allowed estimation of the risk aversion parameter when the asset market equations do not empirically hold exactly, in contrast to previous studies. The method requires specifying the information set used by agents to form forecasts. The covariance parameters that appear as coefficients in asset demand were similar across information sets, although the parameters of risk aversion were somewhat different. But as found in other studies, these risk aversion parameters are imprecisely estimated so that a large range of values cannot be rejected.

Interestingly, the results in this paper and the other studies are similar despite the difference in estimation methods used to identify the covariance restrictions of the international mean-variance portfolio model. The covariance parameter estimates here are roughly like those of Frankel and Engel (1984), although the point estimates of the risk aversion parameter in this paper are closer to zero. At the same time, the model in this paper has relaxed the assumptions that purchasing power parity must hold and that asset market equations contain no disturbances.

Appendix

Derivation of the Model

The utility of a representative investor-consumer in country h is given in equation $\langle 17 \rangle$. The utility function exhibits constant relative risk aversion, ρ with respect to real wealth, W^h.

$$\langle 17 \rangle \qquad U_t^h = U^h(W_t^h; \rho^h),$$

where

$$U_W^h > 0, \qquad U_{WW}^h \leqslant 0$$

and

$$\rho^h = -U_{WW}^h / (W^h \cdot U_W^h),$$

for U_W and U_{WW} the first and second partial derivatives of U with respect to W. Real wealth to each country includes the total holdings of outside assets denominated in the k different currencies. The set of outside assets consists of interest-bearing government debt and the money supply. Since domestic residents purchase consumption goods that are priced in their own currency units, they hold domestic money balances for transaction purposes, but do not hold foreign money balances.

If the h-country residents' holdings of domestic money and i-currency bonds are defined as M^h and B^{hi}, respectively, the vector of asset shares is described below.

$$\langle 18 \rangle \qquad x^h = \begin{vmatrix} B^{h1}/\tilde{W}^h \\ \vdots \\ (B^{hh} + M^h)/\tilde{W}^h \\ \vdots \\ B^{h(k-1)}/\tilde{W}^h \end{vmatrix},$$

where $\tilde{W}^h$ is the nominal wealth of h-country residents.

The investor must then choose x_t to maximize expected future utility at $t+1$, which depends upon the mean and variance of the future value of wealth. Substituting the asset shares given in $\langle 18 \rangle$ into wealth at time $t+1$ (defined in equation $\langle 2 \rangle$ in the text) and then

substituting the result into equation $\langle 17 \rangle$ gives the utility function in terms of rates of return and the asset supplies. Taking the expected value of this form of the utility function and finding the first-order conditions with respect to each component of $\langle 18 \rangle$ gives a set of equations in the conditional mean and variance of wealth. Substituting into these first-order conditions the solutions for the mean and variance, given by equations $\langle 3 \rangle$ and $\langle 4 \rangle$ in the text, implies the following set of equations.

$$\langle 19 \rangle \qquad x_t^b = \rho^{-1} E_t[\Omega^{-1}(r_t - r_t^* \iota - D(s_t))] - E[\Omega^{-1}\Gamma^b],$$

where

$$\Gamma^b = E[\varepsilon_{t+1}^\iota (\pi_{t+1}^b - E_t \pi_{t+1}^b)], \qquad (k-1) \times 1.$$

So asset demand for residents of country b depends upon the covariances between the domestic inflation rate (measured in terms of the numeraire), π^b, and the different nominal exchange rates, $D(s)$.

The demand for assets denominated in different currencies depends upon two components. The first component on the right-hand side of equation $\langle 19 \rangle$ describes the relationship between the nominal relative rates of return, $r - r^* \iota - D(s)$, and the demand for assets, x^b. This relationship depends upon two factors. First, as the conditional variance of the uncertain component in a particular exchange rate rises, demand for the asset declines. Second, if investors are more risk averse, measured by ρ, their demand for assets depends less upon this covariance relationship. The other component of asset demand, $E[\Omega^{-1}\Gamma^b]$, hedges the risk associated with the uncertainty of domestic inflation. Since inflation is not perfectly forecastable, there is no riskless real rate of return. Hence, this component of asset demand is also the portfolio that minimizes the variance of real wealth.

Summing equations $\langle 19 \rangle$ over all countries indexed by b gives the aggregate world demand for outside assets denominated in different currencies. Solving in terms of the relative rates of return by inverting this aggregate portfolio provides equation $\langle 5 \rangle$ in the text.

Statistics

The parameter estimates reported in the text were obtained using the one-step maximum likelihood method. The one-step maximum likelihood estimator takes an initial consistent estimate of the parameter vector, $\hat{\phi}^0$, to the linearized maximum likelihood estimator, $\hat{\phi}^L$, according to:

$$\hat{\phi}^L = \hat{\phi}^0 - [L_{\phi\phi}(\hat{\phi}^0)]^{-1} L_\phi(\hat{\phi}^0)$$

where $L_{\phi\phi}$ is the hessian of the likelihood function and L_ϕ is the gradient vector. This estimator is asymptotically efficient with covariance matrix equal to the information matrix. (See Schmidt, 1976, pp. 234–236 for a derivation of this result.)

To implement this method, the gradient vector must therefore be derived in order to empirically evaluate both the gradient vector itself and the information matrix. Rewriting the system of equations from the text, the likelihood function for the following system must be defined.

$$\begin{bmatrix} r_t - r_t^* \iota \\ D(s_t) \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \theta_1(L) \\ \theta_1(L) \\ \theta_2(L) \end{bmatrix} z_t + \begin{bmatrix} \rho\Omega & \rho\Gamma \\ 0 \end{bmatrix} \begin{bmatrix} x_t \\ w_t \end{bmatrix} + \begin{bmatrix} \psi_t \\ \varepsilon_{t+1}^1 \\ \varepsilon_{t+1}^2 \end{bmatrix},$$

where ι is defined as the $(k-1) \times 1$ vector of ones; and 0 is the $(2k-1) \times (2k-1)$ matrix of zeros.

Ω and Γ are respectively the $(k-1) \times (k-1)$ covariance matrix of exchange rate forecast

errors and the $(k-1) \times k$ covariance matrix of exchange rate and inflation forecast errors. Hence, Ω and Γ can be identified from the partitioned residual covariance matrix.

$$
E \begin{bmatrix} \psi_t \\ \varepsilon^1_{t+1} \\ \varepsilon^2_{t+1} \end{bmatrix} [\psi'_t, \varepsilon^{1\,\prime}_{t+1}, \varepsilon^{2\,\prime}_{t+1}] = Q \quad = \begin{bmatrix} Q_{11} & \vdots & Q_{12} & Q_{13} \\ \cdots & \vdots & \cdots & \cdots \\ Q_{12} & \vdots & \Omega & \Gamma \\ Q_{13} & \vdots & \Gamma' & Q_{33} \end{bmatrix} .
$$

Deriving the gradient then requires maximizing the joint likelihood function subject to this constraint. The log likelihood function is:

$$
L = -\frac{(3k-2)T}{2} \log(2\pi) - \frac{T}{2} \log |Q| - \frac{1}{2} \sum_{t=1}^{T} Y'_t Q^{-1} Y_t ,
$$

where

$$
Y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{bmatrix} ;
$$

for

$$
\begin{aligned}
y_{1t} &= r_t - ir^*_t - \theta_1(L)z_t - \rho\Omega x_t - \rho\Gamma w_t, & (k-1)\times 1, \\
y_{2t} &= D(s_t) - \theta_1(L)z_t, & (k-1)\times 1, \\
y_{3t} &= \pi_{t+1} - \theta_2(L)z_t, & k\times 1.
\end{aligned}
$$

The gradient vector, L_ϕ, is given by the first derivative of the log likelihood function with respect to the parameter vector. Due to the constraints from the covariance matrix, it is useful to write the log likelihood function in terms of the partitioned inverse of the covariance matrix. This is given as:

$$
Q^{-1} = \begin{bmatrix} Q^{11} & \vdots & Q^{12} & Q^{13} \\ \cdots & \vdots & \cdots & \cdots \\ Q^{21} & \vdots & Q^{22} & Q^{23} \\ Q^{31} & \vdots & Q^{32} & Q^{33} \end{bmatrix} .
$$

Following the convention that $Q^{i\cdot} = [Q^{i1} \, Q^{i2} \, Q^{i3}]$ and vice versa for $Q^{\cdot j}$, the derivative with respect to ρ can be written:

$$
L_\rho = \sum_{t=1}^{T} [\Omega x_t + \Gamma w_t]' Q^{1\cdot} Y_t .
$$

The derivatives with respect to the vector regression parameters, $\theta_1(L)$ and $\theta_2(L)$ can also be written in convenient forms:

$$
L_{\theta_1} = \sum_{t=1}^{T} [Q^{1\cdot} + Q^{2\cdot}] Y_t z'_t
$$

and

$$
L_{\theta_2} = \sum_{t=1}^{T} Q^{3\cdot} Y_t z'_t .
$$

The covariance parameters, $\hat{q}_{ij}$, for the elements not contained in Γ and Ω use the relations:

$$\frac{\partial \log |Q|}{\partial Q} = (Q^{-1})'$$

and

$$\frac{\partial (x'Q^{-1}x)}{\partial Q} = -Q^{-1}xx'Q^{-1}.$$

Thus, without the constraints imposed by Ω and Γ, the derivative is given by:

$$L_Q = -\frac{T}{2}Q^{-1} + \frac{1}{2}\sum_{t=1}^{T} Q^{-1} Y_t Y_t' Q^{-1}.$$

However, by the assumption of mean-variance optimization, the Q_{22} and the Q_{23} partitions of Q enter as parameters in the asset market equations as Ω and Γ, respectively. Thus, for covariance parameters that are components of these partitions, the derivatives of the log likelihood function include the following terms:

$$\frac{\partial L}{\partial Q_{22}}: \quad -2\rho Q^{1\cdot} Y_t x_t' \qquad \text{for off-diagonal terms}.$$

$$\frac{\partial L}{\partial Q_{23}}: \quad -2\rho Q^{1\cdot} Y_t w_t'.$$

These relations hold for an arbitrary matrix, Q. But since Q is symmetric, the off diagonal elements also carry a multiple of 2, unlike the diagonal elements.

The estimates in the text were obtained under some simplifying assumptions. First, the structural error in the asset demand equations are uncorrelated with the one-step ahead forecast errors; i.e., $E(\psi_t \varepsilon_{t+1}) = 0$. Also, the θ_1 terms are used to obtain estimates of ε_{t+1} and are not forced to constrain the portfolio equations in subsequent estimation. Finally, although the size of the information matrix is reduced by assuming all of the Q_{12} and Q_{13} terms are equal to zero, the information matrix is still quite large due to the number of covariance terms and coefficients in the vector autoregression. (The number of parameters in the first information set is 208 and in the expanded information set, it is 340.) Therefore, in constructing the parameter estimates and standard errors, it was assumed that:

$$L_{\Gamma_{Q_{33}}} = L_{\Omega Q_{33}} = 0.$$

Inspecting the covariance matrix of the initial estimates indicated that these values were small, so that the assumption appeared reasonable. Under this assumption together with the requirement that $E(\varepsilon_{t+1}\psi_t) = 0$, the information matrix becomes block diagonal. The relevant block for estimating ρ is then the block which contains the parameters: ρ, Ω, Γ, Q_{11}.

Notes

1. For empirical studies that investigate the behavior of the forward prediction error, see for example Cumby and Obstfeld (1984), Hansen and Hodrick (1980), and Hsieh (1984). Empirical studies of the risk premium using intertemporal CAPM investigate the first-order conditions of optimal intertemporal substitution of a representative foreign exchange investor. See, for instance, Cumby (1986), Hodrick and Srivastava (1984), Hansen and Hodrick (1983), and Mark (1985). This approach measures risk relative to an unobserved 'benchmark portfolio' and does not require measures of the market portfolio unlike the static mean-variance approach to be analyzed below.

2. Although this approximation ignores Jensen's inequality, a continuous-time version of the model was also derived and estimated that explicitly identified the Jensen's inequality terms under the assumption that prices and exchange rates follow a diffusion process (see Lewis, 1985). However, including these terms did not affect the overall results. Therefore, the analysis in the text uses the approximation in equation ⟨1⟩ for comparison with the literature.

3. Engel and Rodriguez (1987) have recently estimated the basic model of Frankel and Engel (1984) under purchasing power parity but allowing the conditional variances to vary.

4. Frankel (1982) measures these shares for empirical purposes by calculating the share of imports to GNP for a particular year, 1973. The details of the series construction are given in the data appendix to his paper.

5. In this way, the system of aggregate asset demand equations may be inverted to solve for a single real rate of return common to residents of all countries rather than a combination of the different real returns across countries.

6. A version of the estimation model in the text was also estimated assuming an autoregressive process for ψ_t. Although the parameter estimates of risk aversion were slightly different than those in the text, the autoregressive parameters were all statistically insignificant. Other versions of the theoretical model also gave similar results. See, for example, note 2.

7. The nominal interest rate is the marginal value of money holdings in nominal terms to the investor, an implication of models where money is in the utility function or where money provides services through a transactions technology. See Fama and Farber (1979), for instance. Also, Branson and Henderson (1985) describe in their survey how a bifurcated portfolio decision rule arises under a variety of assumptions regarding the technology of money in consumption. First, the investor decides upon current period consumption and therefore money balances according to the level of the interest rate. Second, given the portfolio and money balances, the individual decides upon the *composition* of the portfolio according to the relative rates of return. The empirical results below combine money together with bonds to form total 'outside assets' as in, for instance, Frankel (1986). However, parameter estimates based upon a portfolio of bonds alone excluding money were not substantially different.

8. The likelihood ratio tests the constraint that all of the coefficients of the vector regressions of a particular lag are zero. If Σ_τ is the covariance matrix of the vector regression at lag length τ, and $\bar{\tau}$ is the maximum assumed lag, then the likelihood ratio test is:

$$T(\log |\Sigma_\tau| - \log |\Sigma_t|) \sim \chi^2((2k-1) \times H \times (\bar{\tau} - \tau)),$$

where $(2k-1)$ is the number of forecasting equations and H is the number of components added to the information set each time period. In this case, $\bar{\tau}$ was chosen as 4. The constraint could not be rejected at lags 4 and 3.

9. To get efficient estimates, all three blocks of the system must be estimated jointly since the covariance matrix from the bottom block is used to identify Ω and Γ.

10. This estimator has the same asymptotic properties as the maximum likelihood estimator. Schmidt (1976, pp. 234–266) proves this result.

11. To keep the exposition clear in the tables, the covariance terms were converted into units appropriate for rates of return in basis point terms. Dividing by 100^2 converts them back into the units relevant for comparing to Frankel and Engel (1984).

12. The Gallant and Jorgensen (1979) test was used to test the constraint. It is the asymptotic equivalent of the likelihood ratio test.

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